6. Non-uniform bending

Introduction

Definition
A non-uniform bending is the case where the cross-section is not only bent but also sheared. It is known from the statics that in such a case, the bending moment in a member is not constant, hence the name “non-uniform bending”. The case takes place as a result of the member loading, perpendicular to its longitudinal axis. This explains an alternative name of the case as a transverse loading. It is obvious that the bending moment creates normal stress in the cross-section, while the shear force creates shearing stresses in that section.

Bernoulli’s hypothesis
We assume that the cross-section, which is a plane before loading, remains a plane after loading as result of the pure bending. However, it is only an approximate solution when compared to real beams. It can be presented by means of simple consideration. The plane cross-section means that the straight lines at the side surface of the beam remain straight. Thus, the shearing strains at the surface would be constant in the cross-section and due to linear physical law (the Hooke’s law). The same refers to the shear stresses. This contradicts the static boundary condition, stating the shear stresses at the extreme fibers should be zero, Fig. 6.1.

Fig. 6.1. Warping due to shear force
Fig. 6.1.a) presents the plane cross-sections before loading and b) after the circular loading (the right angles remain right, there is no shearing strain). The deformation of c) type is not allowed because of non-zero shear strains at extreme fibers. The case d) represents the cross-section warping with maximum value of shear strain at the neutral axis and zero at the extreme fibers. The observation of the real beams’ deformations also shows the warping of the cross-sections. However, the solution of the theory of elasticity shows, that if the shear force is constant, the distribution of normal stresses caused by the bending moment is the same as in the pure (circular) bending, i.e. it is the same as for the plane cross-sections. For the cases of non-constant shear force and common cross-section shapes, the error is quite acceptable1.

Normal stress
In virtue of the Bernoulli’s hypothesis the normal stress formula is similar to that of the pure (circular) bending:

\[ \sigma_x = \frac{M_y(x)}{J_y} z, \]

but the bending moment as well as the stress are functions of the axis coordinate.

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**Shear stress**

As it has been shown, the computation of the shearing stress always requires simplifying hypotheses, which introduce errors, the importance of which depends on the shape of the cross-section. From the equilibrium condition for an element cut from the beam under non-uniform bending, we get the formula for the shearing stress in the cross-section plane:

\[
\tau_{xz} = \frac{Q(x)S'_y(z)}{J_y b(z)},
\]

where:
- \(Q(x)\) is the cross-section shear force,
- \(S'_y(z)\) is the static moment of cut cross-section part,
- \(J_y\) is the inertia moment of the whole cross-section,
- \(b(z)\) is the width of the cut line.

The sense and sign of the shear stress depends on the sense of the cross-section shear force and the assumed sense of the \(x\)-coordinate but the dependence is not as straightforward as it seems. Let’s consider an element cut from a beam with positive shear force, Fig. 6.2.

Positive shear force means the increasing moment with bottom fibers tensioned or the decreasing moment with bottom fibers compressed. In both cases, the element’s balance is ensured by the shearing forces \(T\) of the same sense. These longitudinal shear forces cause in turn the shearing stress, \(\tau_{xz}\), with the same direction and sense. Knowing that the shear stresses act at the edge in pairs, \(\tau_{xz} = \tau_{zx}\), we can determine the sign of the shear stress. The sign is not objective; it depends on the adopted coordinate axes. Usually, the maximum value of the shear stress is attained at the neutral axis for the normal stress.

**Design**

In most cases the dominant criterion in the design of a beam for strength is the maximum value of the normal stress in the beam:

\[
\max |\sigma| = \frac{M_y}{W_y} \leq R.
\]

Next, the condition for the shear stress should be verified:
where  \( R \) is the shearing strength, a material constant.

**Examples**

**Example 6.1**

Determine the shear stress distribution for the cross-sections: a) rectangular, b) triangular (isosceles triangle \( b \times h \)) and c) circular.

**Solution**

a)

We calculate:

\[
\tau_{xz} = \frac{Q \left( \frac{b}{2} \cdot \frac{b}{2} - \frac{z^2}{2} \right)}{bh^3} = \frac{3Q \left( 1 - \left( \frac{z}{h} \right)^2 \right)}{2bh}.
\]

The maximum value is in the neutral axis:

\[
\max(\tau_{xz}) = \tau_{xz} (z = 0) = \frac{3Q}{2A}.
\]

The stress distribution is parabolic with the maximum 50% greater than the cross-section average.

b) width and height as \( z \)-functions:

\[
b(z) = \frac{2}{3} b - \frac{b}{2} z, \quad h(z) = \frac{2}{3} h - z
\]

cross-section geometry term:

\[
\frac{S_s(z)}{b(z)} = \frac{1}{2} h(z) \left( \frac{z}{h} + \frac{h(z)}{3} \right)
\]

we search a cut line with the maximum value of the shear stress:

\[
\frac{\partial \tau_{xz}}{\partial z} = 0 \quad \rightarrow \quad \frac{\partial}{\partial z} \left[ \left( \frac{2}{3} h - z \right) \left( \frac{2}{3} h + \frac{2}{3} z \right) \right] = 0 \quad \rightarrow \quad z = \frac{h}{6}
\]

The maximum value is attained at the middle of the triangle height.

c) static inertia moment:

\[
S_s' (z) = \frac{2r^2}{3} \cos^3 \alpha
\]

width of the cut:

\[
b(z) = 2r \cos \alpha
\]

so:

\[
|\tau_{xz}| = \frac{Q \cdot \frac{2}{3} r^3 \cos^3 \alpha}{\frac{4}{3} \pi r^2 \cdot 2r \cos \alpha} = \frac{4}{3} \cdot \frac{Q}{r^2} \left( 1 - \frac{z^2}{r^2} \right)
\]

and, finally, we find the maximum value at \( z = 0 \):

\[
\max|\tau_{xz}| = \frac{4}{3} \cdot \frac{Q}{A}.
\]

**Example 6.2**

Determine the ratio of maximum values of the normal and shear stress in a cantilever with a rectangular cross-section \( b \times h \), loaded by a point force at its free end \( x = l \).

**Solution**
the maximum value of the bending moment: \( M_{\text{max}} = Pl \)
the shear force: \( Q = P \) (constant)
the maximum value of the normal stress: \( \max|\sigma_x| = \frac{6Pl}{bh^2} \)
the maximum value of the shear stress: \( \max|\tau_c| = 1.5 \frac{P}{bh} \)
the ratio: \( \frac{\max|\sigma_x|}{\max|\tau_c|} = \frac{6Pl}{bh^2 \cdot 1.5P} = \frac{4}{h} \).

As we can see, usually the normal stress is much greater than the shear stress.

**Example 6.3**
The welded profile IPES 600, made by the steelworks “Pokój” has the dimensions: the total height of 600 mm, the width of the flanges: 220 mm, the height of the flanges: 23 mm, the web thickness: 8 mm.
Determine what part of the cross-section shear force does the web carry.

**Solution**

the static moment in the web:
\[
S_y(z) = 1460 + 307 - 0.4z^2 = 1767 - 0.4z^2 \text{ cm}^3
\]
\[
S_y(z = 0) = 1767 \text{ cm}^3
\]
\[
S_y(z = 277) = 1460 \text{ cm}^3
\]
i.e. the diagram of the shear stress in the web is “flat”

the static moment in the flange:
\[
S_y(y) = 2.3 \cdot (11 - y) \cdot 28.85 = 66.36 \cdot (11 - y) \text{ cm}^3
\]
\[
S_y(y = 0) = 730 \text{ cm}^3
\]

In Fig. 6.3 the diagram of the shear stress and the shear stress flow are shown.

![Fig. 6.3 Shear stress in IPES 600](image)

We calculate the shear force part carried by the web:
\[
Q = b \int_{-h/2}^{h/2} \frac{QS_y(z)}{J_y b} dz = \frac{92200}{95600}Q = 0.965Q,
\]
i.e. the web carries 96.5% of the shear force.

**Example 6.4.**
A simply supported beam with a span length \( l = 4\text{m} \) and the rectangular cross-section \( a \times 2a \) is loaded by continuous loading \( q = 90 \text{kN/m} \). Determine the value of parameter \( a \) of the cross-section if the acceptable values of normal and shear stress are \( R = 300 \text{ MPa} \) and \( R_t = 100 \text{ MPa} \), respectively. Draw the principal stresses’ lines.

**Solution**
the maximum value of the bending moment:
\[ \max M_y = \frac{ql^2}{8} = \frac{90 \cdot 10^3 \cdot 4^2}{8} = 180 \text{ kNm} \]

the section modulus:
\[ W_y = \frac{a(2a)^2}{6} = \frac{2}{3} a^2 \]

the design:
\[ \max |\sigma| = \frac{\max M_y}{W_y} \leq R \rightarrow \frac{180 \cdot 10^3 \cdot 3}{2a^3} \leq 300 \cdot 10^6 \rightarrow a \geq 0.0965 \pm 0.1 \text{ m} \]

the verification of the shear stress
the maximum value of the shear force:
\[ \max |Q| = 180 \text{ kN} \]

the shear stress:
\[ \max |\tau_{xz}| = \frac{3}{2} \frac{180 \cdot 10^3}{0.1 \cdot 0.2} = 13.5 \text{ MPa} < R_t \]

The principal stresses:
- in the extreme fibers there is the normal stress only, so the directions of the principal stress are vertical and horizontal,
- in the neutral axis there is the shear stress only, so the directions of principal stress are rotated by 45 degrees
- in other fibers at any \( z \) we have:
\[ \tan \alpha = \frac{\tau}{\sigma}. \]

The principal stress lines are drawn in Fig. 9.4. Two families of the principal stresses’ lines are perpendicular to each other. Trajectories of the tension principal stresses justify the location of reinforcement in RC beam.

**Review problems**

**Problem 6.1**
A timber beam with the span of 3 m and the width 90 mm is to support three concentrated forces shown in Fig. 9.5. Knowing that for the grade of timber used \( \sigma_{acc} = 12 \text{ MPa} \) and \( \tau_{acc} = 0.8 \text{ MPa} \), determine the minimum required depth \( d \) of the beam.
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Fig. 6.5 Beam with load

**Problem 6.2**
For the non-uniform bending of the cross-section of the beam in Fig. 6.6, determine the normal, shear and principal stresses at point K. Assume that all cross-section dimensions are given.

![Fig. 6.6 Beam and cross-section](image)

**Problem 6.3**
A timber beam of a rectangular cross-section carries a single concentrated load $P$ in its midpoint, Fig. 6.7. Determine the depth $h$ and the width $b$ of the beam, knowing that $P = 40$ kN, $R_t = 1.5$ MPa, $R = 12$ MPa.

![Fig. 6.7 Beam with loading](image)

**Addendum**

**Longitudinal shear flow**
The longitudinal shear force ($T$ in Fig. 6.2) per unit length is called the longitudinal shear flow. The importance of this internal force becomes visible when compare a beam with the same beam cut lengthways, Fig. 6.8.

![Fig. 6.8 Beam and beam cut lengthways](image)

In the cut beam the stresses and the curvature are 2 and 4 times greater, respectively.

**Shear centre**
The equilibrium condition requires that the action axis of the shear force has a position which coincides with the line of action of the resultant of the shearing stress. Therefore, the position of the action axis of the shear force is not arbitrary. There are two internal forces introducing shearing stresses in the cross-section: the shear force and the torsional moment.

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2 from: da Silva, op.cit.
Let’s calculate the shear stresses in the channel cross-section in Fig. 6.9. The resultants of the shearing stress in the web ($R_a$) and in the flanges ($R_b$) reduce not only to the shear force but also to the torsion moment. To reduce the torsion moment, the shear force should be applied at some distance $d$ from the web.

![Fig. 6.9 Channel cross-section](image)

A shear centre is a point which has the following property: if the line of action of the shear force passes through this point, it will not induce torsion of the bar.

The thin-walled cross-sections with concurrent and straight wall elements, Fig. 6.10, are a particularly simple case of determination of the shear centre. Because the resultants of the shearing stresses in the different wall elements pass through the intersection of the centre lines, the moment of the shearing stress in relation to this point vanishes, which means that it is the shear centre.

![Fig. 6.10 Shear centre in particular cases.](image)

**Glossary**

- non-uniform bending, transverse loading – zginanie poprzeczne
- longitudinal shear flow – siła rozwarstwiająca
- strength for shear stress – wytrzymałość na ścianie
- RC, reinforced concrete – żelbet
- shear centre – środek ścianania